HOME WORK 4: INVARIANTS, GAMES, ALGEBRA

1. Invariants and games

1. An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. Is it possible to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$ from the triple $(2, \sqrt{2}, \frac{1}{\sqrt{2}})$ using this operation?

2. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1s and four 0s. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

3. Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, 3, ..., n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

2. Algebra

1. Show that for no positive integer n can both $n+3$ and n^2+3n+3 be perfect cubes.

2. Let a and b be coprime integers greater than 1. Prove that for no $n \geq 0$ is $a^{2n} + b^{2n}$ divisible by $a + b$.

3. Prove that any integer can be written as the sum of five perfect cubes.

4. Solve in real numbers the equation

$$\sqrt[3]{x-1} + \sqrt[3]{x} + \sqrt[3]{x+1} = 0.$$